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LETTER TO THE EDITOR

Fractal dimensions in three-dimensional Kauffman cellular automata

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Abstract. The Kauffman random networks of automata are studied on a simple cubic lattice by computer simulations. Each automaton follows random rules, depending on its six neighbours and fixed in time. A transition between the frozen and the chaotic phase is observed and the fractal dimension of the asymptotic actual damage at the phase transition is measured.

Let us consider a system of N spins, or Boolean variables, σ_i, which can take two possible values, either zero or unity, and are placed at the sites of a lattice. The time evolution of the system is determined by a set of Boolean functions, f_i, one for each spin, each depending on K variables σ_j, i.e. K input sites, not necessarily different, chosen among the N automata. There are 2^{2^K} possible Boolean functions of K variables and the functions f_i are randomly chosen among these 2^{2^K} possibilities. For each time t, then the value of the spin σ_i at site i is given by

$$\sigma_i(t+1)=f_i(\sigma_{j_1}(t),\ldots,\sigma_{j_K}(t)). \tag{1}$$

This model was first introduced by Kauffman (1969, 1984, 1986) in order to describe the mechanism leading to the differentiation of the various cell types in a biological system, starting from an ensemble of different genes (or automata) obeying the same random rules.

The interest in these models has increased beyond biology, since they are suitable to describe more general problems, such as the spreading of a single defect in a complex system (Packard and Wolfram 1985, Stanley *et al* 1987, Costa 1987), the transition from an ordered to a disordered phase, and the crucial question of stability of a complicated structure against minor damage.

Depending on the choice of the K variables for the functions f_i, different cases of this model can be defined; in particular, the K input sites can be fixed to be the K neighbours of the given site i on the lattice (finite-dimensional case), or they can be chosen at random among the N spins in the system (infinite-dimensional case). Much recent theoretical and computational work has been done on this model, in the infinite-dimensional case (Derrida and Pomeau 1986, Derrida and Flyvbjerg 1986, Derrida 1986), and in two dimensions (Derrida and Stauffer 1986, Weisbuch and Stauffer 1986, Stauffer 1987).

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The aim of the present letter is to find, to our knowledge for the first time, results in three dimensions and to determine fractal dimensions.

Depending on parameters, there exist two possible phases for a system of N spins: a frozen phase, in which an initial defect introduced in the system will remain confined and may eventually disappear; and a chaotic phase, in which the defect will spread throughout the system (Kauffman 1969, 1984, 1986, Derrida and Stauffer 1986). We want to investigate the fractal properties of damage spreading throughout the system at the phase transition. We consider two identical configurations, obeying the same set of rules. We introduce an initial damage, i.e. we flip the central spin in one configuration, and we let the two configurations evolve in time until the damage (defined as the set of spins differing in the two configurations) touches the boundaries of the lattice. In the frozen phase, the damage does not spread through the lattice but it remains confined to a small cluster of damaged spins, whereas, in the chaotic phase, the damage spreads with an almost constant propagation velocity. At the transition we can define the following quantities (Stauffer 1987)

$$T \sim L^{d_t} \tag{2}$$

$$M_{act} \sim L^{d_{act}} \tag{3}$$

where T is the time for the damage to touch the boundaries and M_{act} is the mass of the damage at time T. These two novel exponents d_t and d_{act} represent respectively how the time for the damage to spread over a distance L goes to infinity as $L \to \infty$, and the fractal dimension of this damage. In the chaotic phase we have $d_t = 1$ and $d_{act} = d$, since the damage in this case is a homogeneous cluster spanning the whole system. In this way the Kauffman model is treated in the language of other growth processes (Herrmann 1986), which are better known at present.

We consider a system of N spins on a simple cubic lattice of linear dimension L. For each spin, we define the rule f_i to depend on its nearest neighbours; since in a simple cubic lattice the number of nearest neighbours is 6, for each of the L^3 spins we need $2^6 = 64$ memory allocations for the function f_i. These 64 Boolean function values are selected equal to unity with probability p and to zero with probability $1-p$ and, once they are defined initially, they are kept constant throughout the simulation. Following Derrida and Stauffer (1986) we study the transition by varying the continuous parameter p. We start with a random configuration of spins; therefore we need to analyse only the range of p between 0 and 0.5, due to the symmetry of the Boolean functions.

In order to save memory, we have used the multispin coding technique, storing one spin per bit, but also storing n rows of the lattice in a computer work. On the Cyber 76, where the computer word has 60 bits, we take $L = 60/n$, with the integer n such that L/n is also an integer and $n < 30$. Then it is possible to store L^3 spins in L^2/n memory allocations. For example, in the first word the rows $1, (L/n)+1, \ldots, (n-1)(L/n)+1$ are stored, this arrangement simplifying the utilisation of periodical boundary conditions.

Starting from an initial configuration of random spins, we let the system evolve in time according to (1) and we update simultaneously all the spins. Our alogrithm allows one spin flip in 2.1 microseconds.

First we determine at what value of the parameter p the phase transition occurs. We start with two identical random configurations of spins, obeying the same set of rules. Then we change the state of a small fraction $\psi(0)$ of spins in one configuration and we let them both evolve in time t. We measure then the damage $\psi(t)$, i.e. the fraction of spins which differ between the two configurations, in the limit $t\to\infty$ (i.e. after a number of the order of 10^3 time steps).

Figure 1 shows our results, as a function of p, for two different system sizes ($L=15$, 30) and two different values of the initial amount of damage. The transition between the frozen and the chaotic phase is seen to occur at $p_c\approx 0.12$. For $p>p_c$ the system is chaotic, $\psi(\infty)$ remains finite and is roughly independent of the initial distance. At $p=0.5$, even if the system is totally random, $\psi(\infty)$ is slightly less than $\frac{1}{2}$ (about 0.49), evidence of the correlation between two configurations because they obey the same rules. As p approaches 0.12, $\psi(\infty)$ goes to zero in the limit $\psi(0)\to 0$. In this sense, $\psi(0)$ and $\psi(\infty)$ are analogous to magnetic field and magnetisation in magnetic phase transitions (Derrida and Stauffer 1986). For large values of $\psi(0)$, there is no transition and $\psi(\infty)$ varies smoothly.

Now we investigate the fractal properties of the spreading damage. We introduce only a single initial defect in the system ($\psi(0)=L^{-3}$) and monitor its spreading in

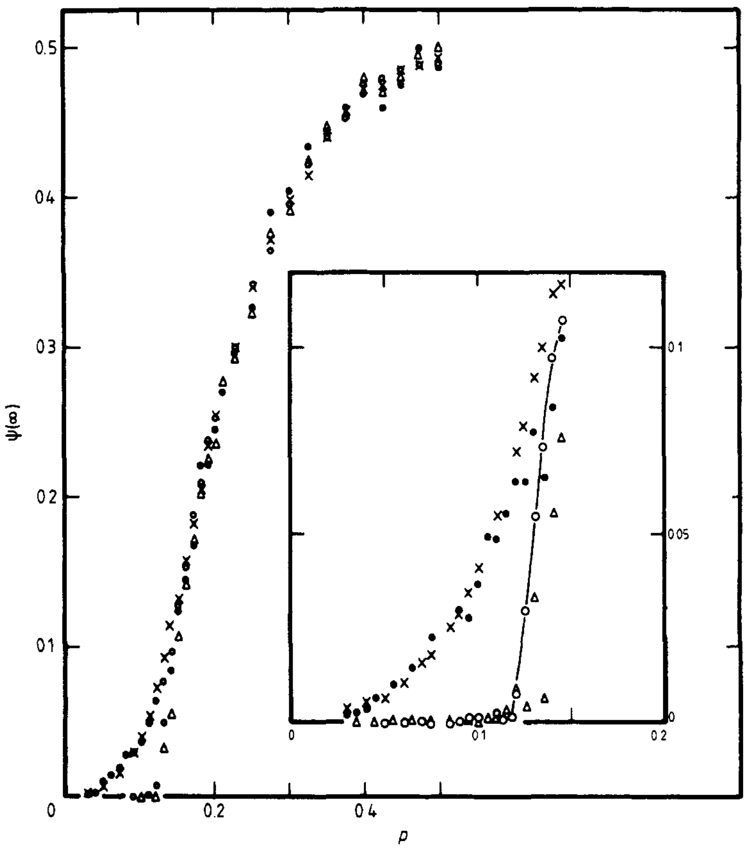

Figure 1. The asymptotic damage $\psi(\infty)$ is plotted as a function of the probability p for the system sizes $L=15$ ($\psi(0)=0.001$, ○ and $\psi(0)=0.104$, ×) and $L=30$ ($\psi(0)=0.001$, △ and $\psi(0)=0.104$, ●). The data are taken after 1000 time steps. The transition occurs at $p\approx 0.12$. In the inset the same quantities are plotted in the vicinity of the transition. The data corresponding to the smaller system size exhibit a transition at a value of p larger than 0.12, due to finite size effect. For the larger value of $\psi(0)$ the data do not exhibit a sharp transition but they go smoothly to zero near $p=0$.

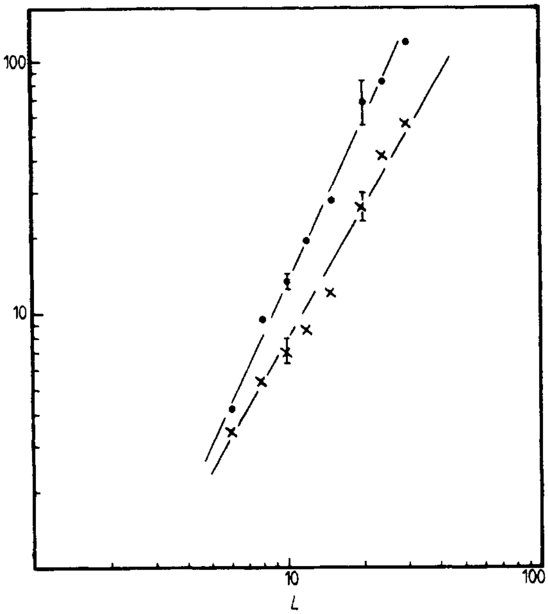

Figure 2. Log-log plot of the average touching time $\langle T \rangle$ (●), and the average mass at the time of touching $\langle M_{act} \rangle$ (×) against L. The average is taken over the runs in which the damage reaches the boundaries of the lattice (e.g. 1500 runs for $L = 6$, 140 runs for $L = 15$, 5 runs for $L = 30$). The slopes are $d_t \approx 2.22$ and $d_{act} \approx 1.77$.

time. At $p \approx 0.12$, we find (figure 2) that

$$d_t \approx 2.2 \qquad \text{and} \qquad d_{act} \approx 1.8 \tag{4}$$

the corresponding values on the square lattice are (Stauffer 1987)

$$d_t \approx 1.7 \qquad \text{and} \qquad d_{act} \approx 1.5. \tag{5}$$

In both the two- and the three- dimensional cases, the fractal dimension d_{act} appears to be significantly lower than the lattice Euclidean dimension.

In conclusion, we have studied the Kauffman model on a simple cubic lattice by computer simulation. We observe a sharp transition between the frozen and the chaotic phase at a value of $p \approx 0.12$. At the transition, we measure the time for an initial defect to touch the boundaries of the lattice and the mass of this damage. These quantities scale with two novel exponents $d_t \approx 2.2$ and $d_{act} \approx 1.8$.

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